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THE POTTS MODEL ON A BETHE LATTICE IN AN EXTERNAL FIELD

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A solution for the Potts model with arbitrary number of states on a Bethe lattice in a nonzero external field has been obtained. A line of first-order phase transitions has been constructed in the temperature – external-field plane, terminating at the point of the second-order phase transition. The magnitude of the magnetization jump on the phase-transition lines has been found, as well as some of the critical exponents characterizing this phase transition.

Keywords: phase transitions, Potts model, critical exponents.

INTRODUCTION

The Potts model [1] is one of the most widely used models in statistical physics and is a theoretical tool that has been applied to a wide class of phenomena in the physics of condensed media [2, 3]. In addition, the Potts model is employed in problems arising in nuclear physics. Numerous exact results for the Potts model have been reported in the literature. It is well known that if the number of spin states in the Potts model is greater than some critical value (which depends on the dimensionality of the lattice), then in the absence of an external field a first-order phase transition is observed, and if it is less, then a second-order phase transition is observed [1, 3, 4]. The phase transition in the Potts model in the absence of an external field has often been considered both for a pure magnet [1–4] and also for a magnet with nonmagnetic impurities [7, 8].

However, the critical behavior of the Potts model in the presence of an external field is also of unquestionable interest [5, 6]. Therefore, in the present work we consider the Potts model with an arbitrary number of states on a Bethe lattice in an external field. And even if the Bethe lattice is only nominally comparable with real crystalline lattices, it possesses the advantage that the Potts model in an external field can be solved in this lattice exactly.

PHASE TRANSITIONS IN THE POTTS MODEL IN AN EXTERNAL FIELD

The Potts model [1] is formulated in the following way. Let us consider some regular lattice. We assign to each node a quantity σ_i (*spin*) which can take s different values, let us say, 1, 2, …, *s*. Two neighboring spins σ_i and σ_j interact with interaction energy $-J_p\delta(\sigma_i,\sigma_j)$, where

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$$-J_p\delta(\sigma_i,\sigma_j)=\begin{cases}1, & \sigma_i=\sigma_j,\\ 0, & \sigma_i\neq\sigma_j.\end{cases}$$

Let there be an external field $H>0$, which acts on state 1. Then the total energy is equal to

$$E=-J_p\sum_{(i,j)}\delta(\sigma_i,\sigma_j)-H\sum_i\delta(\sigma_i,1)\,. \tag{1}$$

The solution of the problem with the Hamiltonian defined by Eq. (1) consists, in particular, in determining the quantities p_i – the fractions (relative numbers) of spins found, in thermal equilibrium, in state i. This problem can be solved in a relatively simple way for the Bethe lattice – a tree without closed paths, by making use of a method described in [1]. We construct a Bethe lattice with arbitrary coordination number q in the following way. Let us consider a *central* node of the lattice with spin σ_0; q neighbors of this node form the first shell, $q-1$ *external* (besides the central node) neighbors of each node of the first shell form the second shell, and so on out to the *N*th shell. Thus

$$p_i=\frac{\sum_{\{\sigma\}}\delta(\sigma_0,i)P(\{\sigma\})}{Z}\,, \tag{2}$$

where

$$Z=\sum_{\{\sigma\}}P(\{\sigma\})\,,\ P(\{\sigma\})=\exp\left(K\sum_{(i,j)}\delta(\sigma_i,\sigma_j)+h\sum_i\delta(\sigma_i,1)\right),$$

$K=J_p/kT$, $h=H/kT$, k is the Boltzmann constant, T is the temperature, and the sum in Eq. (2) is carried out over all possible spin configurations $\{\sigma\}$. We represent $P(\{\sigma\})$ in the form

$$P(\{\sigma\})=\mathrm{e}^{h\delta(\sigma_0,1)}\prod_{j=1}^{q}\exp\left(K\delta(\sigma_0,\sigma_1^{(j)})+K\sum_{(l,n)}\delta(\sigma_l^{(j)},\sigma_n^{(j)})+h\sum_l\delta(\sigma_l^{(j)},1)\right),$$

where $\sigma^{(j)}$ is the set of spins of the *j*th branch. Thus

$$Z=\sum_{\sigma^{(j)}}\mathrm{e}^{h\delta(\sigma_0,1)}\left(G_N(\sigma_0)\right)^q\,,$$

$$G_N(\sigma_0)=\sum_{\sigma^{(j)}}\exp\left(K\delta(\sigma_0,\sigma_1^{(j)})+K\sum_{(l,n)}\delta(\sigma_l^{(j)},\sigma_n^{(j)})+h\sum_l\delta(\sigma_l^{(j)},1)\right). \tag{3}$$

Substituting these expressions in Eq. (2) and introducing the notation $x_{N,k}=G_N(k)/G_N(1)$, ($k=2, 3, ..., s$), we obtain

$$p_1=\frac{\mathrm{e}^h}{\mathrm{e}^h+\sum_k x_{N,k}^q}\,,\ p_j=\frac{x_{N,j}^q}{\mathrm{e}^h+\sum_k x_{N,k}^q}\,,\ j>1\,. \tag{4}$$